Solution to Midterm Examination No. 1

1. (a) Performing forward elimination, we can have

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \overset{\boldsymbol{M}_1}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{matrix} \\ \text{(subtract row 1)} \\ \text{(subtract row 1)} \\ \text{(subtract row 1)} \end{matrix}$$

$$\overset{\boldsymbol{M}_2}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ \\ \text{(subtract row 2)} \\ \text{(subtract row 2)} \end{matrix}$$

$$\overset{\boldsymbol{M}_3}{\Longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \\ \\ \text{(subtract row 3)} \end{matrix} = \boldsymbol{U}$$

where

$$\boldsymbol{M}_1 = \boldsymbol{E}_{41}\boldsymbol{E}_{31}\boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \ \boldsymbol{M}_2 = \boldsymbol{E}_{42}\boldsymbol{E}_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix},$$

and

$$\boldsymbol{M}_3 = \boldsymbol{E}_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$$

Since $\boldsymbol{M}_3\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{A} = \boldsymbol{U}$, we have

$$\boldsymbol{A} = \boldsymbol{M}_1^{-1}\boldsymbol{M}_2^{-1}\boldsymbol{M}_3^{-1}\boldsymbol{U} = \boldsymbol{L}\boldsymbol{U}$$

where

$$\boldsymbol{U} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \boldsymbol{L} &= \boldsymbol{M}_1^{-1}\boldsymbol{M}_2^{-1}\boldsymbol{M}_3^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \end{aligned}$$

(b) From (a), there are 4 nonzero pivots and hence rank($\boldsymbol{A}$) = 4. Therefore, $\boldsymbol{A}$ is invertible. Also from (a),

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U} \Longrightarrow \boldsymbol{A}^{-1} = \boldsymbol{U}^{-1}\boldsymbol{L}^{-1}.$$

Since

$$\boldsymbol{M}_3\boldsymbol{M}_2\boldsymbol{M}_1\boldsymbol{A} = \boldsymbol{L}^{-1}\boldsymbol{A} = \boldsymbol{U}$$

we have

$$\begin{aligned}\boldsymbol{L}^{-1} &= \boldsymbol{M}_3\boldsymbol{M}_2\boldsymbol{M}_1\\ &= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ -1 & 0 & 1 & 0\\ -1 & 0 & 0 & 1\end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0\\ -1 & 1 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & -1 & 1\end{bmatrix}.\end{aligned}$$

We then use the Gauss-Jordan method to find $\boldsymbol{U}^{-1}$:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 & -1\\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & -1 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1\end{array}\right]$$

$$\Longrightarrow \boldsymbol{U}^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Finally, we can obtain

$$\boldsymbol{A}^{-1} = \boldsymbol{U}^{-1}\boldsymbol{L}^{-1} = \begin{bmatrix} 2 & -1 & 0 & 0\\ -1 & 2 & -1 & 0\\ 0 & -1 & 2 & -1\\ 0 & 0 & -1 & 1\end{bmatrix}.$$

(c) From (a), we know that rank($\boldsymbol{A}$) = 4. Hence $\dim(\boldsymbol{C}(\boldsymbol{A}^T)) = 4$.

(d) Since $\boldsymbol{A}$ is invertible, the system is always solvable for all $b_1, b_2, b_3, b_4 \in \mathcal{R}$.

(e) Since rank($\boldsymbol{A}$) $= 4$, we have the dimension of $\mathcal{N}(\boldsymbol{A}^T)$ is $4-4=0$. Therefore, the only vector in $\mathcal{N}(\boldsymbol{A}^T)$ is $\mathbf{0}$.

2. (a) True. Since

$$(\boldsymbol{ABA})^T = \boldsymbol{A}^T\boldsymbol{B}^T\boldsymbol{A}^T = (-\boldsymbol{A})(-\boldsymbol{B})(-\boldsymbol{A}) = -(\boldsymbol{ABA})$$

$\boldsymbol{ABA}$ is also skew-symmetric.

(b) True. Suppose $\boldsymbol{A}$ is m by n. Consider that $\boldsymbol{Ax} = \boldsymbol{b}$ always has at least one solution for every $\boldsymbol{b} \in \mathcal{R}^m$, and we have rank($\boldsymbol{A}$) $= r = m \leq n$. Then $\dim(\boldsymbol{N}(\boldsymbol{A}^T)) = m - r = m - m = 0$, and the only solution to $\boldsymbol{A}^T\boldsymbol{y} = \mathbf{0}$ is $\boldsymbol{y} = \mathbf{0}$.

(c) False. Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

They are both singular matrices in M. Since

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which is nonsigular, the singular matrices in M do not form a subspace of M.

(d) False. Consider $x_1 \cdot (2, 1, -1) + x_2 \cdot (4, 1, 1) + x_3 \cdot (2, -1, 5) = (0, 0, 0)$. We have

$$\left[\begin{array}{ccc|c} 2 & 4 & 2 & 0 \\ 1 & 1 & -1 & 0 \\ -1 & 1 & 5 & 0 \end{array}\right]$$

which by elimination can be reduced to

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence there exist nonzero solutions (x_1, x_2, x_3), which implies that $(2, 1, -1)$, $(4, 1, 1)$, and $(2, -1, 5)$ are not linearly independent. Therefore, they do not form a basis for $\mathcal{R}^3$.

3. (a) Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

We can have $S = \mathcal{N}(\boldsymbol{A})$. The RRE form of $\boldsymbol{A}$ can be given by

$$\boldsymbol{A'} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

The free variables are x_3 and x_4. Letting $(x_3, x_4) = (1, 0)$, we have $(x_1, x_2) = (1, -1)$. Letting $(x_3, x_4) = (0, 1)$, we have $(x_1, x_2) = (0, -1)$. As a result, a basis for S can be given by

$$(1, -1, 1, 0),\ (0, -1, 0, 1).$$

(b) Let

$$\boldsymbol{B} = \left[\begin{array}{cccc} 1 & 1 & 1 & 1 \end{array}\right].$$

We can have $T = \mathcal{N}(\boldsymbol{B})$. The free variables are x_2, x_3, and x_4. Letting $(x_2, x_3, x_4) = (1, 0, 0)$, we have $x_1 = -1$. Letting $(x_2, x_3, x_4) = (0, 1, 0)$, we have $x_1 = -1$. Letting $(x_2, x_3, x_4) = (0, 0, 1)$, we have $x_1 = -1$. As a result, a basis for T can be given

$$(-1, 1, 0, 0),\ (-1, 0, 1, 0),\ (-1, 0, 0, 1).$$

(c) Let

$$\boldsymbol{C} = \left[\begin{array}{c} \boldsymbol{A} \\ \boldsymbol{B} \end{array}\right] = \left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right].$$

We can have

$$\begin{aligned} & S \cap T \\ = & \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_4 = 0, x_2 + x_3 + x_4 = 0, x_1 + x_2 + x_3 + x_4 = 0\} \\ = & \mathcal{N}(\boldsymbol{C}). \end{aligned}$$

Since the RRE form of $\boldsymbol{C}$ is

$$\boldsymbol{C'} = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

the rank of $\boldsymbol{C'}$ is 3. Hence $\dim(S \cap T) = \dim(\mathcal{N}(\boldsymbol{C'})) = 4 - \text{rank}(\boldsymbol{C'}) = 1$.

(d) Assume $\boldsymbol{u} \in S + T$. By the definition of $S + T$, we can have $\boldsymbol{u} = \boldsymbol{s} + \boldsymbol{t}$ where $\boldsymbol{s} \in S$ and $\boldsymbol{t} \in T$. Since we have found a basis for S and T in (a) and (b), respectively, we can express $\boldsymbol{u}$ as

$$\begin{aligned} \boldsymbol{u} = \boldsymbol{s} + \boldsymbol{t} &= a_1(1, -1, 1, 0) + a_2(0, -1, 0, 1) \\ &\quad + b_1(-1, 1, 0, 0) + b_2(-1, 0, 1, 0) + b_3(-1, 0, 0, 1) \end{aligned}$$

where $a_1, a_2, b_1, b_2, b_3 \in \mathcal{R}$. Since $(1, -1, 1, 0)$ is not a linear combination of $\{(-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1)\}$, and $(0, -1, 0, 1) = -(-1, 1, 0, 0) + (-1, 0, 0, 1)$, $\boldsymbol{u}$ can be rewritten as

$$\boldsymbol{u} = a'_1(1, -1, 1, 0) + b'_1(-1, 1, 0, 0) + b'_2(-1, 0, 1, 0) + b'_3(-1, 0, 0, 1)$$

where $a'_1, b'_1, b'_2, b'_3 \in \mathcal{R}$. Since $(1, -1, 1, 0)$, $(-1, 1, 0, 0)$, $(-1, 0, 1, 0)$, and $(-1, 0, 0, 1)$ are linearly independent, they form a basis for $S + T$. As a result, the dimension of $S + T$ is 4.

4. (a) Finding the coefficients x_1, x_2 such that $x_1 \cdot (1, 1, 2) + x_2 \cdot (1, 2, 1) = 0$ is equivalent to finding the solutions of the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

After elimination we can obtain

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

And thus the only solution is $(x_1, x_2) = (0, 0)$, which means that the two vectors $(1, 1, 2)$ and $(1, 2, 1)$ are linearly independent.

(b) Since $1 \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2) + 1 \cdot (\boldsymbol{v}_2 - \boldsymbol{v}_3) + 1 \cdot (\boldsymbol{v}_3 - \boldsymbol{v}_1) = \mathbf{0}$, we know that $\boldsymbol{v}_1 - \boldsymbol{v}_2$, $\boldsymbol{v}_2 - \boldsymbol{v}_3$, and $\boldsymbol{v}_3 - \boldsymbol{v}_1$ are linearly dependent for any $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ in $\mathcal{R}^3$.

(c) Since there are 4 vectors in $\mathcal{R}^3$, they must be linearly dependent.

5. Consider the following augmented matrix and perform elimination:

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 5 & b_2 \\ -1 & -3 & 3 & 0 & b_3 \end{array}\right]$$
$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 1 & b_2 - 2b_1 \\ -1 & -3 & 3 & 0 & b_3 \end{array}\right]$$
$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 1 & b_2 - 2b_1 \\ 0 & 0 & 6 & 2 & b_3 + b_1 \end{array}\right]$$
$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 0 & 0 & 3 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + 5b_1 - 2b_2 \end{array}\right]$$
$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 3b_1 - b_2 \\ 0 & 0 & 3 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 + 5b_1 - 2b_2 \end{array}\right]$$
$$\Longrightarrow \left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 3b_1 - b_2 \\ 0 & 0 & 1 & 1/3 & (1/3) \cdot (b_2 - 2b_1) \\ 0 & 0 & 0 & 0 & b_3 + 5b_1 - 2b_2 \end{array}\right].$$

From the last row, we have that the system is solvable if $b_3 + 5b_1 - 2b_2 = 0$, i.e.,

$$b_3 = 2b_2 - 5b_1.$$

When the above condition holds, we need to solve

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 1 & 3b_1 - b_2 \\ 0 & 0 & 1 & 1/3 & (1/3) \cdot (b_2 - 2b_1) \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The pivot variables are x_1 and x_3, and we can obtain a particular solution

$$\boldsymbol{x}_p = \begin{bmatrix} 3b_1 - b_2 \\ 0 \\ (1/3) \cdot (b_2 - 2b_1) \\ 0 \end{bmatrix}.$$

Now we turn to find the nullspace solution $\boldsymbol{x}_n$. Note that x_2, x_4 are free variables. For $(x_2, x_4) = (1, 0)$, we have $(x_1, x_3) = (-3, 0)$. For $(x_2, x_4) = (0, 1)$, we have $(x_1, x_3) = (-1, -1/3)$. Therefore, the nullspace solution can be given by

$$\boldsymbol{x}_n = x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

where $x_2, x_4 \in \mathcal{R}$. Finally, the complete solution is

$$\boldsymbol{x} = \boldsymbol{x}_p + \boldsymbol{x}_n = \begin{bmatrix} 3b_1 - b_2 \\ 0 \\ (1/3) \cdot (b_2 - 2b_1) \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{bmatrix}$$

where x_2, $x_4 \in \mathcal{R}$ if $b_3 = 2b_2 - 5b_1$.

6. (a) We can know that $\boldsymbol{A}$ must be 3 by 4. Since $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$ is the only solution to $\boldsymbol{Ax} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, the nullspace of $\boldsymbol{A}$ must contain the zero vector only. Hence, the rank of $\boldsymbol{A}$ should be 4. Yet as the number of rows of B is only 3, the rank of $\boldsymbol{A}$ cannot be 4. Therefore, $\boldsymbol{A}$ does not exist.

(b) We can find a desired matrix as follows:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is clear that it is in the RRE form. The pivot variables are x_1, x_2, and x_4, and the free variable is x_3. Taking $x_3 = 1$, we can obtain a special solution as

$$(2, 3, 1, 0).$$

Therefore, the vector $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ forms a basis for the nullspace of the above matrix.