EE 2030 Linear Algebra Chi-chao Chao
Spring 2012

Midterm Examination No. 2
7:00pm to 10:00pm, May 4, 2012

Problems for Solution:

1. (20%) Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

   (a) Find the projection matrix $\boldsymbol{P}$ onto the row space of $\boldsymbol{A}$.

   (b) Find the orthogonal complement of the row space of $\boldsymbol{A}$.

   (c) Given $\boldsymbol{x} = (1, 2, 3)$, split it into $\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$, where $\boldsymbol{x}_r$ is in the row space of $\boldsymbol{A}$ and $\boldsymbol{x}_n$ is in the nullspace of $\boldsymbol{A}$.

   (d) Given

$$\boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

   in the column space of $\boldsymbol{A}$, find a vector $\boldsymbol{x}_r$ in the row space of $\boldsymbol{A}$ such that

$$\boldsymbol{A}\boldsymbol{x}_r = \boldsymbol{b}.$$

2. (10%) Suppose V and W are two subspaces of a given vector space. Recall that the sum of V and W is defined as

$$V + W = \{\boldsymbol{v} + \boldsymbol{w} : \boldsymbol{v} \in V,\ \boldsymbol{w} \in W\}.$$

   If $V \cap W = \{\boldsymbol{0}\}$, then $V + W$ is called the *direct sum* of V and W, with the special notation $V \oplus W$.

   (a) Show that any vector $\boldsymbol{x}$ in the direct sum $V \oplus W$ can be written in one *and only one* way as $\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{w}$ with $\boldsymbol{v} \in V$ and $\boldsymbol{w} \in W$.

   (b) If V is spanned by $(1, 1, 1)$ and $(1, 0, 1)$, find a subspace W so that $V \oplus W = \mathcal{R}^3$.

3. (15%) Let

$$\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R} = \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & a \end{bmatrix}.$$

   (a) Give an orthonormal basis for the column space of $\boldsymbol{A}$.

   (b) For which values of a the rank of $\boldsymbol{A}$ is 2?

(c) Let $a = 2$. Solve $\boldsymbol{Ax} = \boldsymbol{b}$ in the least squares sense for

$$\boldsymbol{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}.$$

4. (10%) Consider the vector space $C[-1, 1]$, the space of all real-valued continuous functions on $[-1, 1]$, with inner product defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

   (a) Find an orthonormal basis for the subspace spanned by 1, x, and x^2.

   (b) Express $2x^2$ as a linear combination of those orthonormal basis functions found in (a).

5. (15%) Find the determinants of

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

6. (20%) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be n by n real matrices. Is each of the following statements true or false? If it is true, prove it. Otherwise, find a counterexample.

   (a) $\det(\boldsymbol{A} + \boldsymbol{B}) = \det \boldsymbol{A} + \det \boldsymbol{B}$.

   (b) If the entries of $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ are all integers, then both determinants are 1 or -1.

   (c) If all the entries of $\boldsymbol{A}$ are integers, and $\det \boldsymbol{A}$ is 1 or -1, then all the entries of $\boldsymbol{A}^{-1}$ are integers.

   (d) If $\boldsymbol{A} \neq \boldsymbol{O}$, but $\boldsymbol{A}^k = \boldsymbol{O}$ (where $\boldsymbol{O}$ denotes the zero matrix) for some positive integer k, then $\boldsymbol{A}$ must be singular.

7. (10%) Suppose the n by n matrix $\boldsymbol{A}_n$ has 3's along its main diagonal and 2's along the diagonal below and the $(1, n)$ position:

$$\boldsymbol{A}_4 = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}.$$

   (a) Find the determinant of $\boldsymbol{A}_4$. (*Hint:* By cofactors of row 1.)

   (b) Find the determinant of $\boldsymbol{A}_n$ for $n > 4$.