Solution to Midterm Examination No. 2

1. (a) We have the projection matrix onto the column space of $\boldsymbol{A}^T$ as

$$\begin{aligned}\boldsymbol{P} &= \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)^{-1}\boldsymbol{A}\\ &= \begin{bmatrix}1&1\\1&0\\1&1\end{bmatrix}\left(\begin{bmatrix}1&1&1\\1&0&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\\1&1\end{bmatrix}\right)^{-1}\begin{bmatrix}1&1&1\\1&0&1\end{bmatrix}\\ &= \begin{bmatrix}1/2&0&1/2\\0&1&0\\1/2&0&1/2\end{bmatrix}.\end{aligned}$$

(b) The orthogonal complement of $\mathcal{C}(\boldsymbol{A}^T)$ is $\mathcal{N}(\boldsymbol{A})$. Since the RRE form of $\boldsymbol{A}$ is

$$\boldsymbol{R}_A = \begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}$$

we can obtain that $(-1,0,1)^T$ is a basis for $\mathcal{N}(\boldsymbol{A})$. As a result, we can have

$$\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{x} : \boldsymbol{x} = x_3(-1,0,1)^T, \forall x_3 \in \mathcal{R}\}.$$

(c) From the projection matrix $\boldsymbol{P}$ derived in (a), we can have

$$\boldsymbol{x}_r = \boldsymbol{P}\boldsymbol{x} = \begin{bmatrix}1/2&0&1/2\\0&1&0\\1/2&0&1/2\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}.$$

And hence

$$\boldsymbol{x}_n = \boldsymbol{x} - \boldsymbol{x}_r = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

(d) We have

$$\left[\begin{array}{ccc|c}1&1&1&2\\1&0&1&3\end{array}\right] \Longrightarrow \left[\begin{array}{ccc|c}1&0&1&3\\0&1&0&-1\end{array}\right].$$

A particular solution $\boldsymbol{x}_p$ to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ can be given by

$$\boldsymbol{x}_p = \begin{bmatrix}3\\-1\\0\end{bmatrix}.$$

Then

$$\begin{aligned}\boldsymbol{x}_r &= \boldsymbol{P}\boldsymbol{x}_p\\ &= \begin{bmatrix}1/2&0&1/2\\0&1&0\\1/2&0&1/2\end{bmatrix}\begin{bmatrix}3\\-1\\0\end{bmatrix} = \begin{bmatrix}3/2\\-1\\3/2\end{bmatrix}.\end{aligned}$$

2. (a) Since $\boldsymbol{x} \in V \oplus W$, we can have $\boldsymbol{x} = \boldsymbol{v}_1 + \boldsymbol{w}_1$ where $\boldsymbol{v}_1 \in V$ and $\boldsymbol{w}_1 \in W$. Suppose there also exist $\boldsymbol{v}_2 \in V$ and $\boldsymbol{w}_2 \in W$ such that $\boldsymbol{x} = \boldsymbol{v}_2 + \boldsymbol{w}_2$. Then we can obtain

$$\begin{aligned} & \boldsymbol{x} = \boldsymbol{v}_1 + \boldsymbol{w}_1 = \boldsymbol{v}_2 + \boldsymbol{w}_2 \\ \Longrightarrow \quad & \boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{w}_2 - \boldsymbol{w}_1. \end{aligned}$$

As $\boldsymbol{v}_1 - \boldsymbol{v}_2 \in V$, $\boldsymbol{w}_2 - \boldsymbol{w}_1 \in W$, and $V \cap W = \{\mathbf{0}\}$, we have $\boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{w}_2 - \boldsymbol{w}_1 = \mathbf{0}$. Therefore, $\boldsymbol{v}_1 = \boldsymbol{v}_2$ and $\boldsymbol{w}_1 = \boldsymbol{w}_2$.

(b) Since $(1, 1, 1)$ and $(1, 0, 1)$ are linearly independent and span V, we have $\dim(V) = 2$. Given $V \cap W = \{\mathbf{0}\}$, we have $\dim(V \oplus W) = \dim(V) + \dim(W)$. Hence $\dim(W) = \dim(V \oplus W) - \dim(V) = \dim(\mathcal{R}^3) - \dim(V) = 3 - 2 = 1$. Let $\boldsymbol{w} = (w_1, w_2, w_3)$ be a basis for W, and it must be independent of $(1, 0, 1)$ and $(1, 1, 1)$. Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

Then the RRE form for $\boldsymbol{A}$ can be found as

$$\boldsymbol{R}_A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & w_3 - w_1 \end{bmatrix}.$$

For the three rows of $\boldsymbol{A}$ to be independent, $\boldsymbol{R}_A$ should have full rank, which implies $w_3 - w_1 \neq 0$. An example for $\boldsymbol{w}$ can be given as $\boldsymbol{w} = (1, 0, 0)$, and W is the subspace spanned by $\boldsymbol{w}$.

3. (a) Let $\boldsymbol{Q} = [\boldsymbol{q}_1 \ \boldsymbol{q}_2 \ \boldsymbol{q}_3]$ where

$$\boldsymbol{q}_1 = \begin{bmatrix} 1/5 \\ 2/5 \\ 2/5 \\ 4/5 \end{bmatrix}, \quad \boldsymbol{q}_2 = \begin{bmatrix} -2/5 \\ 1/5 \\ -4/5 \\ 2/5 \end{bmatrix}, \quad \boldsymbol{q}_3 = \begin{bmatrix} -4/5 \\ 2/5 \\ 2/5 \\ -1/5 \end{bmatrix}.$$

Note that $\boldsymbol{Q}$ has orthonormal columns. If $a \neq 0$, $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$ forms an orthonormal basis for $\mathcal{C}(\boldsymbol{A})$. If $a = 0$, then $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ can do the job.

(b) If $a = 0$, then $\mathcal{C}(\boldsymbol{A})$ is spanned by $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ only, i.e., rank $= 2$

(c) The least squares solution satisfies $\boldsymbol{A}^T\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}$. We can then have

$$
\begin{aligned}
& \boldsymbol{A}^T\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b} \\
\Rightarrow\ & (\boldsymbol{Q}\boldsymbol{R})^T(\boldsymbol{Q}\boldsymbol{R})\hat{\boldsymbol{x}} = (\boldsymbol{Q}\boldsymbol{R})^T\boldsymbol{b} \\
\Rightarrow\ & \boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{R}\hat{\boldsymbol{x}} = \boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{b} \\
\Rightarrow\ & \boldsymbol{R}^T\boldsymbol{R}\hat{\boldsymbol{x}} = \boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{b} && (\because \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}) \\
\Rightarrow\ & (\boldsymbol{R}^T)^{-1}\boldsymbol{R}^T\boldsymbol{R}\hat{\boldsymbol{x}} = (\boldsymbol{R}^T)^{-1}\boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{b} && (\because \text{For } a = 2,\ \boldsymbol{R}^T \text{ is invertible.}) \\
\Rightarrow\ & \boldsymbol{R}\hat{\boldsymbol{x}} = \boldsymbol{Q}^T\boldsymbol{b} \\
\Rightarrow\ & \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 1/5 & 2/5 & 2/5 & 4/5 \\ -2/5 & 1/5 & -4/5 & 2/5 \\ -4/5 & 2/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \\
\Rightarrow\ & \hat{\boldsymbol{x}} = \begin{bmatrix} -2/5 \\ 0 \\ 1 \end{bmatrix}.
\end{aligned}
$$

4. (a) Let $f_1(x) = 1$, $f_2(x) = x$, and $f_3(x) = x^2$. By the Gram-Schmidt process, we can have:

(i) $F_1(x) = f_1(x) = 1, \quad \| F_1(x) \|^2 = \langle F_1(x), F_1(x) \rangle = 2$

$\Longrightarrow q_1(x) = \dfrac{F_1(x)}{\| F_1(x) \|} = \dfrac{1}{\sqrt{2}}.$

(ii) $F_2(x) = f_2(x) - \langle q_1(x), f_2(x) \rangle q_1(x) = x, \quad \| F_2(x) \|^2 = \dfrac{2}{3}$

$\Longrightarrow q_2(x) = \dfrac{F_2(x)}{\| F_2(x) \|} = \sqrt{\dfrac{3}{2}}x.$

(iii) $F_3(x) = f_3(x) - \langle q_1(x), f_3(x) \rangle q_1(x) - \langle q_2(x), f_3(x) \rangle q_2(x) = x^2 - \dfrac{1}{3}$

$\| F_3(x) \|^2 = \dfrac{8}{45} \qquad \Longrightarrow q_3(x) = \dfrac{F_3(x)}{\| F_3(x) \|} = \sqrt{\dfrac{45}{8}}\left(x^2 - \dfrac{1}{3}\right).$

Hence $\{q_1(x), q_2(x), q_3(x)\}$ forms an orthonormal basis for the subspace spanned by 1, x, and x^2.

(b) Since

$$
\begin{aligned}
\langle q_1(x), 2x^2 \rangle &= \frac{2\sqrt{2}}{3} \\
\langle q_2(x), 2x^2 \rangle &= 0 \\
\langle q_3(x), 2x^2 \rangle &= \frac{4\sqrt{10}}{15}
\end{aligned}
$$

we can express $2x^2$ as

$$
\begin{aligned}
2x^2 &= \langle q_1(x), 2x^2 \rangle q_1(x) + \langle q_2(x), 2x^2 \rangle q_2(x) + \langle q_3(x), 2x^2 \rangle q_3(x) \\
&= \frac{2\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} + \frac{4\sqrt{10}}{15} \cdot \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right).
\end{aligned}
$$

5. (a) Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 \end{bmatrix}.$$

Subtracting row 1 from all the other rows, we can have

$$\begin{aligned} \det \boldsymbol{A} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 & 1 \\ 1 & 1 & 4 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \end{vmatrix} \\ = (-1)^2 \cdot 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 = 24. \end{aligned}$$

(b) Let

$$\boldsymbol{B} = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Since the sum of the five rows of $\boldsymbol{B}$ is an all-zero row, we can have $\det \boldsymbol{B} = 0$.

(c) Let

$$\boldsymbol{C} = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

By the cofactor formula, we can have

$$\begin{aligned} \det \boldsymbol{C} &= \begin{vmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{vmatrix} = 3 \begin{vmatrix} 3 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{vmatrix} \\ &= 3 \left\{ 3 \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} \right\} - \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} \\ &= 8 \begin{vmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{vmatrix} \\ &= 8 \left\{ 3 \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} \right\} - 3 \cdot \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} \\ &= 21 \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} - 8 \begin{vmatrix} 1 & 1 \\ 0 & 3 \end{vmatrix} = 21(3 \cdot 3 - 1 \cdot 1) - 8(1 \cdot 3) = 144. \end{aligned}$$

6. (a) *False.*

Let $\boldsymbol{A} = \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det \boldsymbol{A} = \det \boldsymbol{B} = 1$ and $\det(\boldsymbol{A}+\boldsymbol{B}) = 4$. Therefore, $\det(\boldsymbol{A}+\boldsymbol{B}) \neq \det \boldsymbol{A} + \det \boldsymbol{B}$.

(b) *True.*

Since the entries of $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ are all integers, both $\det \boldsymbol{A}$ and $\det \boldsymbol{A}^{-1}$ are integers by the big formula. Also because $\det \boldsymbol{A} \cdot \det \boldsymbol{A}^{-1} = \det \boldsymbol{I} = 1$, both $\det \boldsymbol{A}$ and $\det \boldsymbol{A}^{-1}$ should be 1 or -1.

(c) *True.*

We can have

$$(\boldsymbol{A}^{-1})_{ij} = \frac{C_{ji}}{\det \boldsymbol{A}}$$

where $C_{ij} = (-1)^{i+j} \det \boldsymbol{M}_{ij}$ and $\boldsymbol{M}_{ij}$ is the $(n-1)\times(n-1)$ submatrix of $\boldsymbol{A}$ with row i and column j removed. Since all the entries of $\boldsymbol{A}$ are integers, $\det \boldsymbol{M}_{ij}$ is an integer, and so is C_{ij}. Now because $\det \boldsymbol{A}$ is 1 or -1, $(\boldsymbol{A}^{-1})_{ij}$ is always an integer.

(d) *True.*

Since $\boldsymbol{A}^k = \boldsymbol{O}$ for some positive interger k, we have $\det(\boldsymbol{A}^k) = (\det \boldsymbol{A})^k = 0$. It implies $\det \boldsymbol{A} = 0$, and thus $\boldsymbol{A}$ is singular.

7. (a) Applying the cofactor formula to the first row, we can have

$$\begin{aligned} \det \boldsymbol{A}_4 &= 3\begin{vmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{vmatrix} \\ &= 3^4 - 2^4 = 65. \end{aligned}$$

(b) Applying the cofactor formula to the first row, we can obtain the determinant as a combination of the determinant of an $(n-1)$ by $(n-1)$ lower triangular matrix and that of an $(n-1)$ by $(n-1)$ upper triangular matrix:

$$\begin{aligned} \det \boldsymbol{A}_n &= \begin{vmatrix} 3 & 0 & 0 & \cdots & 0 & 2 \\ 2 & 3 & 0 & \cdots & \cdots & 0 \\ 0 & 2 & 3 & \ddots & & \vdots \\ 0 & 0 & 2 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 2 & 3 \end{vmatrix} \\ &= 3\begin{vmatrix} 3 & 0 & 0 & \cdots & 0 \\ 2 & 3 & 0 & \ddots & \vdots \\ 0 & 2 & 3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 2 & 3 \end{vmatrix} + (-1)^{n+1}\cdot 2\begin{vmatrix} 2 & 3 & 0 & \cdots & 0 \\ 0 & 2 & 3 & \ddots & \vdots \\ 0 & 0 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 3 \\ 0 & \cdots & 0 & 0 & 2 \end{vmatrix} \\ &= 3^n + (-1)^{n+1}\cdot 2^n. \end{aligned}$$