EE 2030 Linear Algebra — Chi-chao Chao
Spring 2013

Final Examination

7:00pm to 10:00pm, June 14, 2013

Problems for Solution:

1. (25%) True or false. (If it is true, prove it. Otherwise, show why it is not or find a counterexample.)

   (a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ is diagonalizable.

   (b) $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} < 0$ for every nonzero vector $\boldsymbol{x}$, where $\boldsymbol{A} = \begin{bmatrix} -4 & 5 & 10 \\ -9 & -10 & -7 \\ -6 & 3 & -5 \end{bmatrix}$ and $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

   (c) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

   (d) The transformation $T : P_3 \to P_3$ defined by $T(p(x)) = x^2 + p(x)$ is linear, where P_3 is the vector space of all real-coefficient polynomials of degree at most 3, i.e., $P_3 = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_0, a_1, a_2, a_3 \in \mathcal{R}\}$.

   (e) $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$, where $\boldsymbol{A}$ is an m by n matrix and $\boldsymbol{A}^+$ is its pseudoinverse.

2. (15%)

   (a) Find the eigenvalues of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   (b) A real matrix $\boldsymbol{A}$ is called skew-symmetric if $\boldsymbol{A}^T = -\boldsymbol{A}$. Show that the eigenvalues of a skew-symmetric matrix are pure imaginary.

   (c) If $\boldsymbol{A}$ is skew-symmetric, show that the quadratic form $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$ for every real vector $\boldsymbol{x}$.

3. (10%) Find the limits as $k \to \infty$ of

$$\begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

4. (10%) Given

$$\boldsymbol{A} = \begin{bmatrix} 2 & 5 \\ -7 & 2 \end{bmatrix}$$

define for every nonzero vector $\boldsymbol{x}$,

$$R(\boldsymbol{x}) = \frac{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}$$

where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Find the minimum of $R(\boldsymbol{x})$, i.e., $\min_{\boldsymbol{x} \neq \boldsymbol{0}} R(\boldsymbol{x})$, and a vector $\boldsymbol{x}$ that achieves the minimum.

5. (15%) Let $M_{2\times 2}$ be the vector space of of all 2 by 2 real matrices and $\mathcal{R}$ be the set of real numbers. The linear transformation $T : M_{2\times 2} \to \mathcal{R}^2$ is defined by

$$T(\boldsymbol{A}) = \boldsymbol{A}\boldsymbol{v}$$

where $\boldsymbol{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   (a) Let $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, which is a basis for $M_{2\times 2}$. Also let γ be the standard basis for $\mathcal{R}^2$. Find the matrix representation $[T]_\beta^\gamma$.

   (b) Find the kernel of T.

   (c) Let $\omega = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, which is a basis for $\mathcal{R}^2$. Find the matrix representation $[T]_\beta^\omega$.

6. (25%) Consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

   (a) Find the singular value decomposition of $\boldsymbol{A}$.

   (b) Find an orthonormal basis for the column space of $\boldsymbol{A}$.

   (c) Is there a left inverse for $\boldsymbol{A}$? If yes, find one.

   (d) Find the shortest leat squares solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, where

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

   (e) Given $\boldsymbol{b}$ in (d), there exist $\boldsymbol{p}$ in the column space of $\boldsymbol{A}$ and $\boldsymbol{e}$ in the left nullspace of $\boldsymbol{A}$ such that $\boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e}$. Find the vector $\boldsymbol{x}_r$ in the row space of $\boldsymbol{A}$ such that $\boldsymbol{A}\boldsymbol{x}_r = \boldsymbol{p}$.