Solution to Final Examination

1. (a) False. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Consider

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = (1-\lambda)^2(2-\lambda) = 0.$$

We can then obtain that the eigenvalues of $\boldsymbol{A}$ are 1, 1, and 2. For $\lambda = 1$, its AM is 2. Since

$$\boldsymbol{A} - 1 \cdot \boldsymbol{I} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

the GM of eigenvalue 1 is 1, which is smaller than its AM. Therefore, $\boldsymbol{A}$ is not diagonalizable.

(b) Let

$$\boldsymbol{B} = \frac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^T\right) = \begin{bmatrix} -4 & -2 & 2 \\ -2 & -10 & -2 \\ 2 & -2 & -5 \end{bmatrix}$$

which is a symmetric matrix. Then we can obtain $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x}$. Hence $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x} < 0$ for every nonzero vector $\boldsymbol{x}$ if and only if $\boldsymbol{B}$ is negative definite, or equivalently,

$$-\boldsymbol{B} = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 10 & 2 \\ -2 & 2 & 5 \end{bmatrix}$$

is positive definite. Since the upper left determinants of $-\boldsymbol{B}$ are 4, 36, 108, which are all positive, $-\boldsymbol{B}$ is positive definite. As a result, $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} < 0$ for every nonzero vector $\boldsymbol{x}$.

(c) True. Since

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1}$$

$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(d) False. Let $p(x) = 1$. Since

$$\begin{aligned} T(cp(x)) &= T(c) = x^2 + c \\ cT(p(x)) &= cT(1) = c(x^2+1) = cx^2 + c \end{aligned}$$

we have $T(cp(x)) \neq cT(p(x))$ as long as $c \neq 1$. Therefore, T is not linear.

(e) True. For every $\boldsymbol{b} \in \mathbb{R}^m$, we can have

$$\boldsymbol{b} = \boldsymbol{p} + \boldsymbol{e}$$

where $\boldsymbol{p} \in \mathcal{C}(\boldsymbol{A})$ and $\boldsymbol{e} \in \mathcal{N}(\boldsymbol{A}^T)$. We can then have, for all $\boldsymbol{b} \in \mathbb{R}^m$,

$$\begin{aligned} \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{b} &= \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+(\boldsymbol{p}+\boldsymbol{e}) \\ &= \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{p} + \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+\boldsymbol{e} \\ &= \boldsymbol{A}^+(\boldsymbol{p}) + \boldsymbol{A}^+\boldsymbol{A}\boldsymbol{0} \\ &= \boldsymbol{A}^+\boldsymbol{p} \\ &= \boldsymbol{A}^+(\boldsymbol{p}+\boldsymbol{e}) \\ &= \boldsymbol{A}^+\boldsymbol{b} \end{aligned}$$

since $\boldsymbol{A}\boldsymbol{A}^+$ is the projection matrix onto $\mathcal{C}(\boldsymbol{A})$ and $\boldsymbol{A}^+\boldsymbol{e} = \boldsymbol{0}$. Therefore, $\boldsymbol{A}^+\boldsymbol{A}\boldsymbol{A}^+ = \boldsymbol{A}^+$.

2. (a) Consider

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0.$$

Hence, the eigenvalues of $\boldsymbol{A}$ are i and $-i$.

(b) Let λ be an eigenvalue of $\boldsymbol{A}$ and $\boldsymbol{x}$ be the corresponding unit eigenvector. Since $\boldsymbol{A}$ is real, we can have

$$\begin{aligned} & \boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \\ \Longrightarrow\ & \bar{\boldsymbol{A}}\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}} \\ \Longrightarrow\ & \boldsymbol{A}\bar{\boldsymbol{x}} = \bar{\lambda}\bar{\boldsymbol{x}}. \end{aligned}$$

Consider $\bar{\boldsymbol{x}}^T\boldsymbol{A}\boldsymbol{x}$. We can then have:

(i) $\bar{\boldsymbol{x}}^T\boldsymbol{A}\boldsymbol{x} = \bar{\boldsymbol{x}}^T(\lambda\boldsymbol{x}) = \lambda\bar{\boldsymbol{x}}^T\boldsymbol{x} = \lambda\|\boldsymbol{x}\|^2 = \lambda$

(ii) $\bar{\boldsymbol{x}}^T\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{A}^T\bar{\boldsymbol{x}})^T\boldsymbol{x} = (-\boldsymbol{A}\bar{\boldsymbol{x}})^T\boldsymbol{x} = (-\bar{\lambda}\bar{\boldsymbol{x}})^T\boldsymbol{x} = -\bar{\lambda}\bar{\boldsymbol{x}}^T\boldsymbol{x} = -\bar{\lambda}\|\boldsymbol{x}\|^2$
$= -\bar{\lambda}$.

Hence, we can obtain $\lambda = -\bar{\lambda}$, which means that λ is pure imaginary.

(c) We can have

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = (\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x})^T = \boldsymbol{x}^T\boldsymbol{A}^T\boldsymbol{x} = \boldsymbol{x}^T(-\boldsymbol{A})\boldsymbol{x} = -\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}.$$

Hence, $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = 0$ for every real vector $\boldsymbol{x}$.

3. Let $\boldsymbol{A} = \begin{bmatrix} 0.4 & 0.2 \\ 0.6 & 0.8 \end{bmatrix}$. Since the eigenvalues of $\boldsymbol{A}$ are 1, 0.2, and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are their corresponding eigenvectors, respectively, we can have

$$\boldsymbol{A} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}$$

where

$$\boldsymbol{S} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Hence, we can have

$$\begin{aligned} \boldsymbol{A}^k = \boldsymbol{S}\boldsymbol{\Lambda}^k\boldsymbol{S}^{-1} &= \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 0.2^k \end{bmatrix} \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & -1/4 \end{bmatrix} \\ \Longrightarrow \lim_{k\to\infty} \boldsymbol{A}^k &= \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & -1/4 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 \\ 3/4 & 3/4 \end{bmatrix}. \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{k\to\infty} \boldsymbol{A}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} \\ \lim_{k\to\infty} \boldsymbol{A}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}. \end{aligned}$$

4. Let

$$\boldsymbol{B} = \frac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^T\right) = \frac{1}{2}\left(\begin{bmatrix} 2 & 5 \\ -7 & 2 \end{bmatrix} + \begin{bmatrix} 2 & -7 \\ 5 & 2 \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

which is a symmetric matrix, and $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x}$. The eigenvalues of $\boldsymbol{B}$ can be found as:

$$\begin{aligned} \det(\boldsymbol{B} - \lambda \boldsymbol{I}) &= \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0 \\ \Longrightarrow \quad & \lambda = 1 \text{ or } 3. \end{aligned}$$

From class, we can have

$$\lambda_{\min} \le R(\boldsymbol{x}) = \frac{\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}} = \frac{\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{x}}{\boldsymbol{x}^T\boldsymbol{x}} \le \lambda_{\max}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of $\boldsymbol{B}$, respectively. Since $\lambda_{\text{mmin}} = 1$, we have

$$\min_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{x}) = 1.$$

Besides, $\boldsymbol{x}$ achieves $\min_{\boldsymbol{x}\neq\boldsymbol{0}} R(\boldsymbol{x})$ if $\boldsymbol{x}$ belongs to the eigenspace corresponding to $\lambda_{\min}$. Since

$$\boldsymbol{B} - \lambda_{\min}\boldsymbol{I} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

we can choose $\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or any nonzero scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

5. (a) We have $\gamma = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, where

$$\boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let $\beta = \{\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3, \boldsymbol{V}_4\}$, where

$$\boldsymbol{V}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{V}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{V}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \boldsymbol{V}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since

$$\begin{aligned} T(\boldsymbol{V}_1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \cdot \boldsymbol{e}_1 + 0 \cdot \boldsymbol{e}_2 \\ T(\boldsymbol{V}_2) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \cdot \boldsymbol{e}_1 + 0 \cdot \boldsymbol{e}_2 \\ T(\boldsymbol{V}_3) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \cdot \boldsymbol{e}_1 + 1 \cdot \boldsymbol{e}_2 \\ T(\boldsymbol{V}_4) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 0 \cdot \boldsymbol{e}_1 + 3 \cdot \boldsymbol{e}_2 \end{aligned}$$

we can have

$$[T]_\beta^\gamma = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

(b) From (a), we can find that $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix} \right\}$ forms a basis for $\mathcal{N}\left([T]_\beta^\gamma\right)$.

Therefore, the kernel of T is given by the span of $-3\boldsymbol{V}_1 + \boldsymbol{V}_2$ and $-3\boldsymbol{V}_3 + \boldsymbol{V}_4$.

(c) Let $\omega = \{\boldsymbol{w}_1, \boldsymbol{w}_2\}$, where

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{w}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Since

$$\begin{aligned} T(\boldsymbol{V}_1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \cdot \boldsymbol{w}_1 + (-1) \cdot \boldsymbol{w}_2 \\ T(\boldsymbol{V}_2) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 6 \cdot \boldsymbol{w}_1 + (-3) \cdot \boldsymbol{w}_2 \\ T(\boldsymbol{V}_3) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-1) \cdot \boldsymbol{w}_1 + 1 \cdot \boldsymbol{w}_2 \\ T(\boldsymbol{V}_4) &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = (-3) \cdot \boldsymbol{w}_1 + 3 \cdot \boldsymbol{w}_2 \end{aligned}$$

we can have

$$[T]_\beta^\omega = \begin{bmatrix} 2 & 6 & -1 & -3 \\ -1 & -3 & 1 & 3 \end{bmatrix}.$$

6. (a) Perform the singular value decomposition, and we can have

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^T = \begin{bmatrix} -\sqrt{6}/6 & -\sqrt{2}/2 & \sqrt{3}/3 \\ \sqrt{6}/3 & 0 & \sqrt{3}/3 \\ -\sqrt{6}/6 & \sqrt{2}/2 & \sqrt{3}/3 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}.$$

(b) Since there are 2 nonzero singular values of $\boldsymbol{A}$, the rank of $\boldsymbol{A}$ is 2. The dimension of the column space of $\boldsymbol{A}$ is 2, and an orthonormal basis for the column space of $\boldsymbol{A}$ can be obtained as the first two columns of $\boldsymbol{U}$, i.e.,

$$\begin{bmatrix} -\sqrt{6}/6 \\ \sqrt{6}/3 \\ -\sqrt{6}/6 \end{bmatrix}, \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}.$$

(c) Since $\boldsymbol{A}$ has full column rank, there is a left inverse for $\boldsymbol{A}$. We can have

$$\boldsymbol{A}^+\boldsymbol{A} = \boldsymbol{I}$$

and hence the pseudoinverse

$$\begin{aligned} \boldsymbol{A}^+ &= \boldsymbol{V\Sigma}^+\boldsymbol{U}^T \\ &= \begin{bmatrix} -2/3 & 1/3 & 1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \end{aligned}$$

is a left inverse for $\boldsymbol{A}$.

(d) The shortest least squares solution is

$$\boldsymbol{A}^+\boldsymbol{b} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}.$$

(e) We can have

$$\boldsymbol{x}_r = \boldsymbol{A}^+\boldsymbol{b} = \begin{bmatrix} 1/3 \\ -1/3 \end{bmatrix}.$$