EE 2030 Linear Algebra — Chi-chao Chao

Spring 2013

Midterm Examination No. 1

7:00pm to 10:00pm, March 29, 2013

Problems for Solution:

1. (a) (5%) Solve $\boldsymbol{Ax} = \boldsymbol{b}$ by solving two triangular systems $\boldsymbol{Lc} = \boldsymbol{b}$ and $\boldsymbol{Ux} = \boldsymbol{c}$:

$$\boldsymbol{A} = \boldsymbol{LU} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

   (b) (5%) Is $\boldsymbol{A}$ in (a) invertible? If yes, find the third column of its inverse.

2. (10%) Find the $\boldsymbol{PA} = \boldsymbol{LDU}$ factorization for

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & -3 & 2 & -2 \\ -1 & 2 & -2 & 1 \end{bmatrix}$$

   where $\boldsymbol{P}$ is a permutation matrix, $\boldsymbol{L}$ is a lower triangular matrix with unit diagonal, $\boldsymbol{D}$ is a diagonal matrix, and $\boldsymbol{U}$ is an upper triangular matrix with unit diagonal.

3. (15%) True or false. (If it is true, prove it. Otherwise, find a counterexample.)

   (a) (5%) Let $\boldsymbol{C}$ be an n by n matrix. Then $(\boldsymbol{I} + \boldsymbol{C})(\boldsymbol{I} - \boldsymbol{C}^T)$ is a symmetric matrix, where $\boldsymbol{I}$ is the identity matrix.

   (b) (5%) Let S and T be subspaces of a vector space V. Then $S \cap T$ is a subspace of V.

   (c) (5%) Let $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and $\boldsymbol{x}_3$ be linearly independent vectors in $\mathcal{R}^4$, where $\mathcal{R}$ is the set of real numbers, and let $\boldsymbol{A}$ be a nonsingular 4 by 4 matrix. If $\boldsymbol{y}_1 = \boldsymbol{Ax}_1$, $\boldsymbol{y}_2 = \boldsymbol{Ax}_2$, and $\boldsymbol{y}_3 = \boldsymbol{Ax}_3$, then $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, and $\boldsymbol{y}_3$ are linearly independent.

4. (10%) Let M denote the vector space of all 3 by 2 real matrices. Is each of the following subsets of M actually a subspace? If yes, prove it and *find the dimension*. Otherwise, find a counterexample.

   (a) (5%) All 3 by 2 matrices with full column rank.

   (b) (5%) All 3 by 2 matrices with the sum of all 6 components in the matrix equal to zero.

5. (10%) Write down a matrix $\boldsymbol{A}$ with the required property or explain why no such matrix exists.

(a) (5%) The only solution to $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(b) (5%) A 3 by 2 matrix $\boldsymbol{A}$ for which $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ has no solution and $\boldsymbol{Ax} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has exactly one solution.

6. (15%) Given the vectors

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$$\boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(a) (5%) Are $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_4$ linearly independent in $\mathcal{R}^3$? Explain.

(b) (5%) Do $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ span $\mathcal{R}^3$? Explain.

(c) (5%) Do $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_4$ form a basis for $\mathcal{R}^3$? Explain.

7. (a) (5%) Find column vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ so that $\boldsymbol{A} = \boldsymbol{uv}^T$:

$$\boldsymbol{A} = \begin{bmatrix} 1 & -4 & 2 & 5 \\ 3 & -12 & 6 & 15 \\ -2 & 8 & -4 & -10 \end{bmatrix}.$$

(b) (5%) Find a basis for the row space of $\boldsymbol{A}$.

(c) (5%) Find a basis for the left nullspace of $\boldsymbol{A}$.

8. (15%) Suppose the matrices in $\boldsymbol{PA} = \boldsymbol{LU}$ are

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & -3 & 2 \\ 2 & -1 & 4 & 2 & 1 \\ 4 & -2 & 9 & 1 & 4 \\ 2 & -1 & 5 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 4 & 2 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) (5%) Find a basis for the column space of $\boldsymbol{A}$.

(b) (5%) *True or false:* Rows 1, 2, 3 of $\boldsymbol{A}$ are linearly independent. (You need to explain your result.)

(c) (5%) Find the general solution to $\boldsymbol{Ax} = \boldsymbol{0}$.