Solution to Midterm Examination No. 1

1. (a) First, we solve $\boldsymbol{c}$ from $\boldsymbol{Lc} = \boldsymbol{b}$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Next, we solve $\boldsymbol{x}$ from $\boldsymbol{Ux} = \boldsymbol{c}$:

$$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}.$$

(b) Yes, $\boldsymbol{A}$ is invertible because it has a full set of pivots. Let the third column of $\boldsymbol{A}^{-1}$ be $\hat{\boldsymbol{x}}$. Since $\boldsymbol{A}\hat{\boldsymbol{x}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ which is the same system as that in (a), we can have $\hat{\boldsymbol{x}} = \boldsymbol{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$.

2. First do row exchange as

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & -3 & 2 & -2 \\ -1 & 2 & -2 & 1 \end{bmatrix} \overset{\boldsymbol{P}_{32}}{\Longrightarrow} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & -3 & 2 & -2 \\ 0 & 0 & 1 & -1 \\ -1 & 2 & -2 & 1 \end{bmatrix} = \boldsymbol{PA}$$

and then perform elimination as

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & -3 & 2 & -2 \\ 0 & 0 & 1 & -1 \\ -1 & 2 & -2 & 1 \end{bmatrix} \overset{\boldsymbol{E}_{41}\boldsymbol{E}_{21}}{\Longrightarrow} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\overset{\boldsymbol{E}_{42}}{\Longrightarrow} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \boldsymbol{DU}.$$

Then we have

$$\boldsymbol{E}_{42}\boldsymbol{E}_{41}\boldsymbol{E}_{21}(\boldsymbol{PA}) = \boldsymbol{U}$$

where

$$\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{E}_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{E}_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and } \boldsymbol{E}_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

We can have

$$\boldsymbol{L} = \boldsymbol{E}_{21}^{-1}\boldsymbol{E}_{41}^{-1}\boldsymbol{E}_{42}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}.$$

The factorization $\boldsymbol{PA} = \boldsymbol{LDU}$ is hence given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 2 & -3 & 2 & -2 \\ -1 & 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. (a) False. Let $\boldsymbol{C} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then

$$(\boldsymbol{I} + \boldsymbol{C})\left(\boldsymbol{I} - \boldsymbol{C}^T\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

is not a symmetric matrix.

(b) True. Take $\boldsymbol{x}, \boldsymbol{y} \in S \cap T$ and hence $\boldsymbol{x}, \boldsymbol{y} \in S$, $\boldsymbol{x}, \boldsymbol{y} \in T$. We check the following two cases:

(i) Consider $\boldsymbol{x} + \boldsymbol{y}$. Since $\boldsymbol{x} \in S$ and $\boldsymbol{y} \in S$, we have $\boldsymbol{x} + \boldsymbol{y} \in S$. Similarly, we can obtain $\boldsymbol{x} + \boldsymbol{y} \in T$. Hence $\boldsymbol{x} + \boldsymbol{y} \in S \cap T$.

(ii) Consider $c\boldsymbol{x}$. Since $\boldsymbol{x} \in S$, we have $c\boldsymbol{x} \in S$. Similarly, we can have $c\boldsymbol{x} \in T$. Hence $c\boldsymbol{x} \in S \cap T$.

As a result, $S \cap T$ is a subspace of V.

(c) True. Consider $a\boldsymbol{y}_1 + b\boldsymbol{y}_2 + c\boldsymbol{y}_3 = \mathbf{0}$. We can have

$$a\boldsymbol{y}_1 + b\boldsymbol{y}_2 + c\boldsymbol{y}_3 \;=\; a\boldsymbol{A}\boldsymbol{x}_1 + b\boldsymbol{A}\boldsymbol{x}_2 + c\boldsymbol{A}\boldsymbol{x}_3 \;=\; \boldsymbol{A}\left(a\boldsymbol{x}_1 + b\boldsymbol{x}_2 + c\boldsymbol{x}_3\right) \;=\; \mathbf{0}.$$

Since $\boldsymbol{A}$ is nonsingular, the only solution to the above system is

$$a\boldsymbol{x}_1 + b\boldsymbol{x}_2 + c\boldsymbol{x}_3 = \mathbf{0}.$$

As $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ are linearly independent, $a\boldsymbol{x}_1 + b\boldsymbol{x}_2 + c\boldsymbol{x}_3 = \mathbf{0}$ only if $a = b = c = 0$. Hence $\boldsymbol{y}_1$, $\boldsymbol{y}_2$, $\boldsymbol{y}_3$ are linearly independent.

4. (a) No. Let $V = \{3 \text{ by } 2 \text{ matrices with full column rank}\}$. Consider

$$\boldsymbol{w}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{w}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \in V.$$

Since

$$\boldsymbol{w}_1 + \boldsymbol{w}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \notin V$$

V is not a subspace of M.

(b) Yes. Let

$$W = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} : a_{11} + a_{12} + a_{21} + a_{22} + a_{31} + a_{32} = 0 \right\}.$$

Suppose

$$\boldsymbol{w}_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad \boldsymbol{w}_2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \in W.$$

(i) Consider

$$\boldsymbol{w}_1 + \boldsymbol{w}_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{bmatrix}.$$

Since

$$(a_{11} + b_{11}) + (a_{12} + b_{12}) + (a_{21} + b_{21}) + (a_{22} + b_{22}) + (a_{31} + b_{31}) + (a_{32} + b_{32})$$
$$= (a_{11} + a_{12} + a_{21} + a_{22} + a_{31} + a_{32}) + (b_{11} + b_{12} + b_{21} + b_{22} + b_{31} + b_{32}) = 0$$

we have $\boldsymbol{w}_1 + \boldsymbol{w}_2 \in W$.

(ii) Consider

$$c\boldsymbol{w}_1 = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \\ ca_{31} & ca_{32} \end{bmatrix}.$$

Since

$$\begin{aligned} & ca_{11} + ca_{12} + ca_{21} + ca_{22} + ca_{31} + ca_{32} \\ = \; & c(a_{11} + a_{12} + a_{21} + a_{22} + a_{31} + a_{32}) = 0 \end{aligned}$$

we have $c\boldsymbol{w}_1 \in W$.

Hence M is a subspace of M. Since $a_{32} = -a_{11} - a_{12} - a_{21} - a_{22} - a_{31}$ we can have

$$\boldsymbol{w}_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & -a_{11} - a_{12} - a_{21} - a_{22} - a_{31} \end{bmatrix}$$

$$= a_{11} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} + a_{12} \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} + a_{21} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} + a_{22} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} + a_{31} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

One can check that

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

is a basis for W, and hence the dimension of W is 5.

5. (a) We can know that $\boldsymbol{A}$ must be 2 by 3. Since $\boldsymbol{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is the only solution to $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, the nullspace of $\boldsymbol{A}$ must contain the zero vector only. Hence the rank of $\boldsymbol{A}$ should be 3. Yet as the number of rows of $\boldsymbol{A}$ is only 2, the rank of $\boldsymbol{A}$ cannot be 3. Therefore, $\boldsymbol{A}$ does not exist.

(b) Since $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has exactly one solution, the nullspace of $\boldsymbol{A}$ must contain the zero vector only. Hence $\boldsymbol{A}$ is a matrix with full column rank. And since $\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ has no solution, the third equation corresponding to the RRE form of $\left[\begin{array}{c|c} \boldsymbol{A} & \begin{matrix} 1 \\ 2 \\ 1 \end{matrix} \end{array} \right]$ should be inconsistent. One example of $\boldsymbol{A}$ is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

6. (a) No. Since there are 4 vectors in $\mathcal{R}^3$, they must be linearly dependent.

(b) No. Consider solving

$$a\boldsymbol{x}_1 + b\boldsymbol{x}_2 + c\boldsymbol{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \implies \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 3 & 5 & 0 \\ 2 & 3 & 5 & 1 \end{array}\right] \implies \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

Since the 3rd equation is inconsistent, this system is not solvable. Hence $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$ do not span $\mathcal{R}^3$.

(c) Yes. Consider $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_4 \end{bmatrix}$. Since the RRE form of $\boldsymbol{A}$ is an identity matrix, $\boldsymbol{A}$ has a full set of pivots and $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_4$ are linearly independent. Hence $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_4$ form a basis for $\mathcal{R}^3$.

7. (a) We can obtain

$$\boldsymbol{A} = \begin{bmatrix} 1 & -4 & 2 & 5 \\ 3 & -12 & 6 & 15 \\ -2 & 8 & -4 & -10 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 2 & 5 \end{bmatrix} = \boldsymbol{u}\boldsymbol{v}^T.$$

(b) The matrix $\boldsymbol{A}$ is a rank-one matrix with pivot row $(1, -4, 2, 5)$. Therefore, a basis for the row space of $\boldsymbol{A}$ is $(1, -4, 2, 5)$.

(c) Since $(1, 3, -2)^T$ is the pivot column of $\boldsymbol{A}$, for the left nullspace, we have

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \boldsymbol{0}^T.$$

Therefore, a basis for the left nullspace of $\boldsymbol{A}$ can be given by

$$(-3, 1, 0), \ (2, 0, 1).$$

8. (a) Since the permutation matrix $\boldsymbol{P}$ does not change the order of the columns of $\boldsymbol{A}$, from $\boldsymbol{U}$ we can find that the pivot columns of $\boldsymbol{A}$ are the 1st, 3rd, and 5th columns. Hence a basis for the column space of $\boldsymbol{A}$ can be given by

$$\begin{bmatrix} 0 \\ 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 5 \end{bmatrix}.$$

(b) False. Form the 1st, 2nd, and 4th rows of $\boldsymbol{L}$, we can find that the 1st, 2nd, and 4th rows of $\boldsymbol{PA}$ are linearly dependent. Form $\boldsymbol{P}$, we also know that the 1st, 2nd, and 4th rows of $\boldsymbol{PA}$ are equal to the 2nd, 1st, and 3rd rows of $\boldsymbol{A}$, respectively. Therefore, rows 1, 2, 3 of $\boldsymbol{A}$ are linearly dependent.

(c) The nullspace of $\boldsymbol{A}$ is equal to the nullspace of $\boldsymbol{PA}$. From $\boldsymbol{U}$, we know that x_1, x_3, and x_5 are pivot variables and x_2 and x_4 are free variables. Therefore, the general solution to $\boldsymbol{Ax} = \boldsymbol{0}$ can be given by

$$x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$