EE 2030 Linear Algebra Chi-chao Chao
Spring 2013

Midterm Examination No. 2
7:00pm to 10:00pm, May 3, 2013

Problems for Solution:

1. (20%) Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}.$$

   (a) Find the projection matrix $\boldsymbol{P}$ onto the row space of $\boldsymbol{A}$ and the projection matrix $\boldsymbol{Q}$ onto the nullspace of $\boldsymbol{A}$.

   (b) Find $\boldsymbol{P}+\boldsymbol{Q}$. Explain your result.

   (c) Find $\boldsymbol{PQ}$. Explain your result.

   (d) Show that $\boldsymbol{P}-\boldsymbol{Q}$ is its own inverse. Why?

2. (15%) Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & -6 \\ 3 & 6 \\ 4 & 8 \\ 5 & 0 \\ 7 & 8 \end{bmatrix}.$$

   (a) Find an orthonormal basis for the column space of $\boldsymbol{A}$.

   (b) Write $\boldsymbol{A}$ as $\boldsymbol{QR}$, where $\boldsymbol{Q}$ has orthonormal columns and $\boldsymbol{R}$ is upper triangular.

   (c) Find the least squares solution to $\boldsymbol{Ax}=\boldsymbol{b}$ if

$$\boldsymbol{b} = \begin{bmatrix} -3 \\ 7 \\ 1 \\ 0 \\ 4 \end{bmatrix}.$$

3. (15%) Consider the vector space $C[0,1]$, the space of all real-valued continuous functions on $[0,1]$, with inner product defined by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx.$$

   (a) Show that $u_1(x)=1$ and $u_2(x)=2x-1$ are orthogonal.

(b) Determine $\|u_1(x)\|$ and $\|u_2(x)\|$.

(c) Find the best least squares approximation to $h(x) = \sqrt{x}$ by a linear function.

4. (20%) Find the determinants of

(a) the 4 by 4 symmetric Pascal matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix};$$

(b) the n by n matrix $\boldsymbol{A}$ with entries $a_{ij} = i + j$, for $1 \leq i, j \leq n$;

(c) the 5 by 5 tridiagonal $-1, 1, 2$ matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix};$$

(d) the 8 by 8 Haar matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

(You can use the fact that its determinant is positive.)

5. (20%) True or false. If it is true, prove it. Otherwise, find a counterexample. Assume that all the given matrices are n by n.

(a) $\det(\boldsymbol{I} + \boldsymbol{A}) = 1 + \det \boldsymbol{A}$, where $\boldsymbol{I}$ is the identity matrix.

(b) If $\boldsymbol{x}$ and $\boldsymbol{y}$ are distinct vectors in $\mathcal{R}^n$, i.e., $\boldsymbol{x} \neq \boldsymbol{y}$, and $\boldsymbol{A}$ is a matrix with the property that $\boldsymbol{Ax} = \boldsymbol{Ay}$, then $\det \boldsymbol{A} = 0$.

(c) If $\boldsymbol{B} = \boldsymbol{S}^{-1}\boldsymbol{AS}$ for some nonsingular matrix $\boldsymbol{S}$, then $\det \boldsymbol{A} = \det \boldsymbol{B}$.

(d) If $\boldsymbol{C}$ is the cofactor matrix of a nonsingular matrix $\boldsymbol{A}$, then $\det \boldsymbol{C} = (\det \boldsymbol{A})^{n-1}$.

6. (10%) If $\boldsymbol{A}$ is a nonsingular n by n matrix, show that there must be some permutation matrix $\boldsymbol{P}$ for which $\boldsymbol{PA}$ has no zeros on its main diagonal. It is *not* the $\boldsymbol{P}$ from elimination. (*Hint:* Consider the big formula for the determinant.)