Solution to Midterm Examination No. 2

1. (a) We can obtain the RRE form of $\boldsymbol{A}$ as

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivot rows are the first, second, and third rows, these three rows constitute a basis for $\mathcal{C}(\boldsymbol{A}^T)$. Let

$$\hat{\boldsymbol{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then we can obtain the projection matrix $\boldsymbol{P}$ onto $\mathcal{C}(\boldsymbol{A}^T)$ as

$$\boldsymbol{P} = \hat{\boldsymbol{A}}^T \left(\hat{\boldsymbol{A}}\hat{\boldsymbol{A}}^T\right)^{-1} \hat{\boldsymbol{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

To find the projection matrix $\boldsymbol{Q}$ onto $\mathcal{N}(\boldsymbol{A})$, we first find a basis for $\mathcal{N}(\boldsymbol{A})$. Since the free column of $\boldsymbol{R}$ is the fourth column, we can find the special solution by letting $x_4 = 1$, and then $x_1 = 0$, $x_2 = 0$, $x_3 = -1$. Hence a basis for $\mathcal{N}(\boldsymbol{A})$ is $\left\{ \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$. Let $\tilde{\boldsymbol{A}} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. Then we can obtain the projection matrix $\boldsymbol{Q}$ onto $\mathcal{N}(\boldsymbol{A})$ as

$$\boldsymbol{Q} = \tilde{\boldsymbol{A}} \left(\tilde{\boldsymbol{A}}^T \tilde{\boldsymbol{A}}\right)^{-1} \tilde{\boldsymbol{A}}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix}.$$

(b) From (a), we can have

$$\boldsymbol{P} + \boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is exactly the identity matrix. The reason is as follows. For any $\boldsymbol{x} \in \mathcal{R}^4$, we can decompose it into

$$\boldsymbol{x} = \boldsymbol{x}_r + \boldsymbol{x}_n$$

where $\boldsymbol{x}_r \in \mathcal{C}(\boldsymbol{A}^T)$ and $\boldsymbol{x}_n \in \mathcal{N}(\boldsymbol{A})$. Since $\mathcal{C}(\boldsymbol{A}^T)$ and $\mathcal{N}(\boldsymbol{A})$ are orthogonal, we can have

$$\boldsymbol{P}\boldsymbol{x}_n = \boldsymbol{0} \qquad \text{and} \qquad \boldsymbol{Q}\boldsymbol{x}_r = \boldsymbol{0}.$$

Hence, we can have

$$(\boldsymbol{P}+\boldsymbol{Q})\boldsymbol{x} = (\boldsymbol{P}+\boldsymbol{Q})\boldsymbol{x}_r + (\boldsymbol{P}+\boldsymbol{Q})\boldsymbol{x}_n = \boldsymbol{P}\boldsymbol{x}_r + \boldsymbol{Q}\boldsymbol{x}_n = \boldsymbol{x}_r + \boldsymbol{x}_n = \boldsymbol{x}$$

which means

$$\boldsymbol{P}+\boldsymbol{Q} = \boldsymbol{I}.$$

(c) We can have

$$\boldsymbol{PQ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathcal{C}(\boldsymbol{A}^T)$ and $\mathcal{N}(\boldsymbol{A})$ are orthogonal, we can have

$$\boldsymbol{PQx} = \boldsymbol{PQ}(\boldsymbol{x}_r + \boldsymbol{x}_n) = \boldsymbol{P}(\boldsymbol{Q}\boldsymbol{x}_r + \boldsymbol{Q}\boldsymbol{x}_n) = \boldsymbol{P}\boldsymbol{x}_n = \boldsymbol{0}$$

for any $\boldsymbol{x} \in \mathcal{R}^4$. Hence we can obtain $\boldsymbol{PQ} = \boldsymbol{O}$, the zero matrix.

(d) We can have

$$\boldsymbol{P}-\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

which is a row-exchanging matrix (exchanging rows 3 and 4) and is its own inverse. In general, we can have

$$\begin{aligned} & (\boldsymbol{P}-\boldsymbol{Q})(\boldsymbol{P}-\boldsymbol{Q}) \\ =\ & \boldsymbol{P}^2 - \boldsymbol{PQ} - \boldsymbol{QP} + \boldsymbol{Q}^2 \\ =\ & \boldsymbol{P}^2 - \boldsymbol{O} - \boldsymbol{O} + \boldsymbol{Q}^2 && (\text{ since } \boldsymbol{PQ} = \boldsymbol{QP} = \boldsymbol{O}\ ) \\ =\ & \boldsymbol{P} + \boldsymbol{Q} && (\text{ since } \boldsymbol{P} \text{ and } \boldsymbol{Q} \text{ are projection matrices} \\ & && \quad \text{and hence } \boldsymbol{P}^2 = \boldsymbol{P} \text{ and } \boldsymbol{Q}^2 = \boldsymbol{Q}\ ) \\ =\ & \boldsymbol{I}. \end{aligned}$$

Therefore, $\boldsymbol{P}-\boldsymbol{Q}$ is its own inverse.

2. (a) Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix}$, where

$$\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}, \quad \boldsymbol{a}_2 = \begin{bmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{bmatrix}.$$

By Gram-Schmidt process, we can have

$$\boldsymbol{A}_1 = \boldsymbol{a}_1 = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{q}_1 = \frac{\boldsymbol{A}_1}{\|\boldsymbol{A}_1\|} = \frac{1}{10}\begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \boldsymbol{a}_2 - \left(\boldsymbol{q}_1^T \boldsymbol{a}_2\right)\boldsymbol{q}_1 = \begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{q}_2 = \frac{\boldsymbol{A}_2}{\|\boldsymbol{A}_2\|} = \frac{1}{10}\begin{bmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{bmatrix}.$$

Hence, $\{\boldsymbol{q}_1, \boldsymbol{q}_2\}$ is an orthonormal basis for the column space of $\boldsymbol{A}$.

(b) From (a), we can have

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix}\begin{bmatrix} \boldsymbol{q}_1^T\boldsymbol{a}_1 & \boldsymbol{q}_1^T\boldsymbol{a}_2 \\ 0 & \boldsymbol{q}_2^T\boldsymbol{a}_2 \end{bmatrix} = \boldsymbol{Q}\boldsymbol{R}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} 1/10 & -7/10 \\ 3/10 & 3/10 \\ 2/5 & 2/5 \\ 1/2 & -1/2 \\ 7/10 & 1/10 \end{bmatrix}, \quad \boldsymbol{R} = \begin{bmatrix} 10 & 10 \\ 0 & 10 \end{bmatrix}.$$

(c) The least squares solution $\hat{\boldsymbol{x}}$ is given by

$$\boldsymbol{A}^T\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{A}^T\boldsymbol{b}.$$

Since $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$, we can then have

$$\boldsymbol{R}\hat{\boldsymbol{x}} = \boldsymbol{Q}^T\boldsymbol{b}$$

which gives

$$\begin{bmatrix} 10 & 10 \\ 0 & 10 \end{bmatrix}\hat{\boldsymbol{x}} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

Therefore,

$$\hat{\boldsymbol{x}} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}.$$

3. (a) The inner product $\langle u_1(x), u_2(x)\rangle$ is given by

$$\int_0^1 u_1(x)\, u_2(x)\, dx = \int_0^1 1 \cdot (2x-1)\, dx = \int_0^1 (2x-1)\, dx = 0.$$

Hence, $u_1(x)$ and $u_2(x)$ are orthogonal.

(b) We can have

$$\|u_1(x)\|^2 = \int_0^1 1 \cdot 1 dx = \int_0^1 1 dx = 1$$
$$\|u_2(x)\|^2 = \int_0^1 (2x-1)\cdot(2x-1)\,dx = \int_0^1 \left(4x^2 - 4x + 1\right) dx = \frac{1}{3}.$$

Therefore, $\|u_1(x)\| = 1$ and $\|u_2(x)\| = 1/\sqrt{3}$.

(c) Let $q_1(x) = u_1(x)/\|u_1(x)\| = 1$ and $q_2(x) = u_2(x)/\|u_2(x)\| = \sqrt{3}(2x-1)$; then $q_1(x)$ and $q_2(x)$ are orthonormal. The best least squares approximation to $h(x)$ by a linear function is hence given by

$$\begin{aligned}\hat{h}(x) &= \langle q_1(x), h(x)\rangle\, q_1(x) + \langle q_2(x), h(x)\rangle\, q_2(x) \\ &= \frac{2}{3}\cdot 1 + \frac{2}{15}\sqrt{3}\cdot\sqrt{3}(2x-1) \\ &= \frac{4}{5}x + \frac{4}{15}\end{aligned}$$

since

$$\langle q_1(x), h(x)\rangle = \int_0^1 \sqrt{x}dx = \frac{2}{3}$$

and

$$\langle q_2(x), h(x)\rangle = \int_0^1 \sqrt{x}\sqrt{3}(2x-1)dx = \frac{2}{15}\sqrt{3}.$$

4. (a) We can have

$$\begin{aligned}\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} \\ &= 1\cdot 1\cdot 1\cdot 1 = 1.\end{aligned}$$

(b) For $n = 1$, we have $\det(\boldsymbol{A}) = |2| = 2$. For $n = 2$, we have $\det(\boldsymbol{A}) = \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = -1$. For $n \geq 3$, consider any three consecutive rows in $\boldsymbol{A}$:

$$\begin{matrix} \\ \text{row } i \\ \text{row } i+1 \\ \text{row } i+2 \\ \\ \end{matrix} \begin{bmatrix} & & \vdots & \\ i+1 & i+2 & \cdots & i+n \\ (i+1)+1 & (i+1)+2 & \cdots & (i+1)+n \\ (i+2)+1 & (i+2)+2 & \cdots & (i+2)+n \\ & & \vdots & \end{bmatrix}.$$

Since $2\times$ (row $i+1$) $=$ (row i) $+$ (row $i+2$), the rows of $\boldsymbol{A}$ are dependent. Therefore, $\boldsymbol{A}$ is singular and $\det(\boldsymbol{A}) = 0$.

(c) Let $\boldsymbol{A}_n$ be the n by n tridiagonal -1, 2, 2 matrix and $B_n = \det(\boldsymbol{A}_n)$. We can then have

$$B_n = B_{n-1} + 2B_{n-2}.$$

Since $B_1 = 1$ and $B_2 = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 3$, we can obtain $B_3 = B_2 + 2B_1 = 5$ and $B_4 = B_3 + 2B_2 = 11$. Finally, the desired determinant is $B_5 = B_4 + 2B_3 = 21$.

(d) Since

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}^T$$
$$= \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

the rows of this matrix are mutually orthogonal. It is a hypercube and the absolute value of the volume is the product of the lengths of the row vectors. Since the determinant is known to be positive, it is hence given by

$$\sqrt{8} \cdot \sqrt{8} \cdot \sqrt{4} \cdot \sqrt{4} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} = 128.$$

5. (a) False. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We can have $\det(\boldsymbol{I} + \boldsymbol{A}) = \det \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 4$, which is not equal to $1 + \det(\boldsymbol{A}) = 1 + 1 = 2$.

(b) True. We can have

$$\begin{aligned} & \boldsymbol{Ax} = \boldsymbol{Ay} \\ \Longrightarrow\ & \boldsymbol{Ax} - \boldsymbol{Ay} = \boldsymbol{0} \\ \Longrightarrow\ & \boldsymbol{A}(\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{0}. \end{aligned}$$

Since $\boldsymbol{x} - \boldsymbol{y} \neq \boldsymbol{0}$, $\boldsymbol{A}$ is singular, which gives $\det(\boldsymbol{A}) = 0$.

(c) True. We multiply $\boldsymbol{S}$ on both sides of the equality, which gives

$$\begin{aligned} & \boldsymbol{SB} = \boldsymbol{AS} \\ \Longrightarrow\ & \det(\boldsymbol{SB}) = \det(\boldsymbol{AS}) \\ \Longrightarrow\ & \det(\boldsymbol{S})\det(\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{S}) \\ \Longrightarrow\ & \det(\boldsymbol{B}) = \det(\boldsymbol{A}) \end{aligned}$$

because $\boldsymbol{S}$ is nonsingular and $\det(\boldsymbol{S}) \neq 0$.

(d) True. Since $\boldsymbol{A}$ is nonsingular and $\det(\boldsymbol{A}) \neq 0$, we can have

$$\begin{aligned} & \boldsymbol{A}^{-1} = \frac{\boldsymbol{C}^T}{\det(\boldsymbol{A})} \\ \Longrightarrow\ & \boldsymbol{C}^T = \det(\boldsymbol{A}) \cdot \boldsymbol{A}^{-1} \\ \Longrightarrow\ & \det\left(\boldsymbol{C}^T\right) = (\det(\boldsymbol{A}))^n (\det \boldsymbol{A})^{-1} \\ \Longrightarrow\ & \det(\boldsymbol{C}) = (\det(\boldsymbol{A}))^{n-1}. \end{aligned}$$

6. The big formula states that the determinant of $\boldsymbol{A}$ is the sum of $n!$ simple determinants, times 1 or -1, and every simple determinant chooses one entry from each row and column. Since $\boldsymbol{A}$ is a nonsingular matrix, $\det(\boldsymbol{A}) \neq 0$. It follows that there exists at least one simple determinant of $\boldsymbol{A}$ avoiding all the zero entries in $\boldsymbol{A}$. Therefore, we can find a corresponding permutation matrix $\boldsymbol{P}$ for which the main diagonal of $\boldsymbol{PA}$ is composed of all the nonzero entries in that simple determinant.