EECS 2060 Discrete Mathematics Chi-chao Chao
Spring 2021

Homework Assignment No. 2
Due 10:10am, April 7, 2021

Reading: Grimaldi: Sections 5.1 Cartesian Products and Relations, 7.1 Relations Revisited: Properties of Relations, 7.3 Partial Orders: Hasse Diagrams, 7.4 Equivalence Relations and Partitions, 1.1 The Rules of Sum and Product, 1.2 Permutations, 1.3 Combinations: The Binomial Theorem, 1.4 Combinations with Repetition, 8.1 The Principle of Inclusion and Exclusion, 8.3 Derangements: Nothing Is in Its Right Place.

Problems for Solution:

1. Let $A = \{a, b, c, d\}$. Determine if each of the following relations on A is an equivalence relation. If it is, find the corresponding equivalence classes.
   (a) $R_1 = \{(a, a), (a, b), (a, c), (b, b), (c, b), (c, c), (d, b), (d, c), (d, d)\}$.
   (b) $R_2 = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c), (d, d)\}$.
   (c) $R_3 = \{(a, a), (b, b), (c, c), (d, d)\}$.
2. Determine if each of the relations in Problem 1 is a partial order. If it is, draw the corresponding Hasse diagram.
3. Let $A = \{a, b, c, d\}$.
   (a) How many different relations are there on A?
   (b) How many relations on A are equivalence relations?
4. For $A = \{a, b, c, d, e\}$, $\boldsymbol{M}$ is the zero-one matrix for a partial order R on A:

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

   Here the rows (from top to bottom) and the columns (for left to right) are indexed in the order a, b, c, d, e.
   (a) Find the maximal elements.
   (b) Find a greatest element, if it exists.
   (c) Find a least element, if it exists.
   (d) Find all upper bounds of $\{d, e\}$.
   (e) Find the greatest lower bound of $\{a, b, c\}$, if it exists.

5. (a) In how many ways can eight policemen be divided into four teams of two and sent to patrol areas A, B, C, D?

   (b) Find the number of nonnegative integer solutions to the pair of equations:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 15$$

$$x_2 + x_4 + x_6 = 5.$$

6. (a) Show that

$$n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}.$$

   (*Hint:* Expand $(1+x)^n$ by Binomial Theorem and then take derivative.)

   (b) Use (a) to show that

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}.$$

   (c) Give a combinatorial proof that

$$n2^{n-1} = \sum_{k=1}^{n} k \binom{n}{k}.$$

   (*Hint:* Count in two ways the number of ways to select a committee and to then select a leader of the committee.)

7. Recall that *Euler's phi function* (or called *Euler's totient function*) $\phi(n)$ is defined as the number of integers m in the range $1 \leq m \leq n$ such that m and n are relatively prime, i.e., $\gcd(m, n) = 1$. Find a formula for $\phi(n)$, $n \geq 2$. (*Hint:* Factor n as the product of prime powers, i.e., $n = \prod_{i=1}^{t} p_i^{e_i}$, where p_i's are distinct primes and $e_i \geq 1$, $i = 1, 2, \ldots, t$.)

8. Recall that a permutation of $\mathcal{N}_n = \{1, 2, \ldots, n\}$ is a bijection from $\mathcal{N}_n$ to $\mathcal{N}_n$. For example, a permutation of $\mathcal{N}_3$ can be given by the function π defined as:

$$\pi(1) = 2, \quad \pi(2) = 3, \quad \pi(3) = 1.$$

   A *derangement* of $\mathcal{N}_n$ is a permutation π with the extra property that $\pi(i) \neq i$ for all $i \in \mathcal{N}_n$. For instance, the above example of a permutation of $\mathcal{N}_3$ is a derangement. Denote as d_n the total number of derangements of $\mathcal{N}_n$. Find a formula for d_n. (*Hint:* Let S_n denote the set of all permutations of $\mathcal{N}_n$. Define $A_i = \{\pi \in S_n : \pi(i) = i\}$, for $i = 1, 2, \ldots, n$. Express the set of all derangements of $\mathcal{N}_n$ in terms of A_i, $i = 1, 2, \ldots, n$.)

Homework Collaboration Policy: I allow and encourage discussion or collaboration on the homework. However, you are expected to write up your own solution and understand what you turn in. Late homework is subject to a penalty of 5% to 40% of your total points.