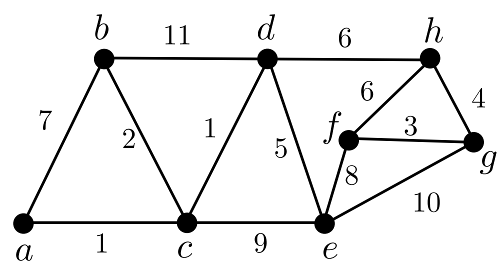
EECS 2060 Discrete Mathematics Chi-chao Chao
Spring 2021

Homework Assignment No. 6
Due 5:00pm, June 23, 2021

Reading: Grimaldi: Sections 13.1 Dijkstra's Shortest-Path Algorithm, 13.2 Minimal Spanning Trees: The Algorithms of Kruskal and Prim, 13.3 Transport Networks: The Max-Flow Min-Cut Theorem; McEliece: Section 4.3 The Matching Problem and the Hungarian Algorithm; Biggs: Sections 17.4 Matchings, 17.5 Maximum Matchings.

Problems for Solution:

1. Consider the weighted simple graph given below.



   (a) Use Dijkstra's algorithm to find a shortest path from vertex b to vertex g.

   (b) Use Dijkstra's algorithm to find a tree of shortest paths from vertex a to all the other vertices.

2. Prove or disprove (with a counterexample) the following for a weighted connected undirected simple graph $G = (V, E)$, where $V = \{v_0, v_1, \ldots, v_n\}$ and $e_1 \in E$ with $w(e_1) < w(e)$ for all $e \in E$, $e \neq e_1$. If Dijkstra's algorithm is applied to G to find the shortest path from v_0 to each of the other vertices, then there exists a vertex v_j, for some $1 \leq j \leq n$, where the edge e_1 is used in the shortest path from v_0 to v_j.

3. Apply Prim's algorithm to find a minimal spanning tree for the weighted graph in Problem 1.

4. (a) A *maximal spanning tree* is a spanning tree whose total weight is as large as possible. Show how to modify Kruskal's algorithm to find a maximal spanning tree.

   (b) Apply your algorithm obtained in (a) to find a maximal spanning tree for the weighted graph in Problem 1.

5. Suppose, in the example taught in class of arranging dance partners at a party with eight girls and eight boys, that each girl likes *exactly four* boys, and that each boy is liked by *exactly four* girls. Without knowing anything else about the graph, show that a complete matching is possible.

6. If $S_1, S_2, \ldots, S_n$ are n subsets of a finite set X, a *system of distinct representatives* (SDR) from them is a list $(s_1, s_2, \ldots, s_n)$ of n distinct elements of X, with $s_i \in S_i$, $i = 1, 2, \ldots, n$. For example, if $S_1 = \{1, 2\}$, $S_2 = \{2, 3\}$, $S_3 = \{1, 2, 3\}$, then $(1, 3, 2)$ is an SDR. For each of the following families of sets, find an SDR or explain why none exists.

   (a) $\{1, 6\}$, $\{4, 5\}$, $\{2, 6\}$, $\{1, 2, 4\}$, $\{1, 6\}$.

   (b) $\{a, m\}$, $\{a, r, e\}$, $\{m, a, e\}$, $\{m, s, t, e, r\}$, $\{m, e\}$, $\{r, a, m\}$.

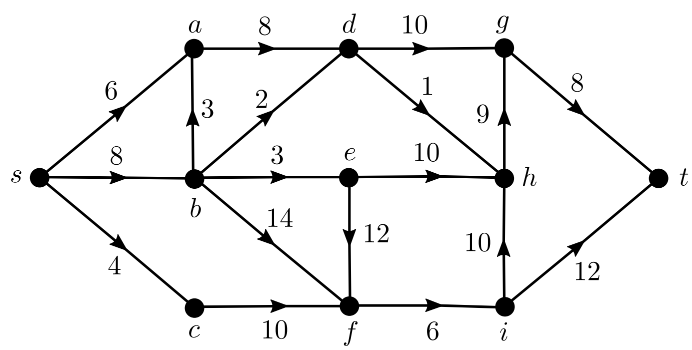
7. Consider the network with vertex set $\{s, a, b, c, t\}$ and edges and capacities given by the following table.

(s,a)	(s,b)	(a,b)	(a,c)	(a,t)	(b,c)	(c,t)
5	2	3	1	3	3	4

   (a) Calculate the capacities of all cuts separating s and t.

   (b) Find the maximum value of a flow from s to t.

8. Consider the following network with source s and sink t. The capacities are shown on the edges.



   Suppose a flow f is given by

$$f(s, a) = f(a, d) = f(d, g) = f(g, t) = 6$$

   with all other $f(x, y) = 0$.

   (a) What is the value of the flow f?

   (b) Find a maximum flow for this network. What is its value?

   (c) Find a minimum cut for this network.

Homework Collaboration Policy: You can discuss the homework problems with any number of students currently taking the course, the teaching assistants, and the instructor. However, solutions should not be exchanged. You should make sure that you understand what you turn in, and should of course write up every word of the solution by yourself. It is OK to compare your final answer with others currently enrolled in the course, but you should fix up any error by your own effort. If these sentences are still vague, just tell yourself *"I shall not take unfair advantage of any other student"* and this should answer other policy-related questions you have in your mind. Late homework is subject to a penalty of 5% to 40% of your total points.